STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

ZW 1962 - 024

On a common fixed point of a commutative transformation semigroup of continuous mappings

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1962

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ZW 1962-024

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The aim of this remark is to prove the theorems 1 and 2. We introduce notation and prove a few lemmas.

Throughout this remark Y will denote a compact topological space, and G will be a commutative semigroup of continuous transformations of Y. The operation in G is assumed to be the composition of mappings:

$$g_1 \circ g_2(y) = g_1 [g_2(y)] .$$

By transformation we mean, as usual, a mapping from a set into itself. Moreover, we shall assume that the identity mapping belongs to G.
If $G' \subset G$, $Y' \subset Y$, then by $G'(Y')$ we denote the set

$$G'(Y') = \{ y : y = g'(y'),\ g' \in G' \quad y' \in Y' \} .$$

If $G' = \{g'\}$ or $Y' = \{y'\}$ then $g'(Y')$ and $G'(y)$ are written instead of $\{g'\}(Y')$ and $G'(\{y'\})$. The set $G(y)$ is called the orbit of y under G. If $G(Y') \subset Y'$ for $Y' \subset Y$, then Y' is said to be invariant under G.

A topological space Y is said to have the fixed point property, or f.p.p., if every continuous transformation of Y has a fixed point.

If Y' is invariant under G, then by $G|Y'$ we denote, as usual, the semigroup G restricted to Y'.

Lemma 1. Let $G(e)=Y$ for some $e \in Y$. Then

$$Z = \bigcap_{g \in G} g(Y) \neq \emptyset \quad .$$

Moreover, Z is invariant under G.

Proof:

The sets $g(Y)$, $g \in G$, are closed, as they are continuous images of a compact space. They also have the finite intersection property, as

$$g_1 \circ g_2 \circ \ldots \circ g_n(e) \in \bigcap_{i=1}^{n} g_i(Y) \quad .$$

Hence, $Z \neq \emptyset$.
Z is an invariant set, as it is the intersection of invariant sets.

Lemma 2. $Z = \bigcap_{y \in Y} G(y)$.

Proof:

$g(Y)=g(G(e))=G(g(e))=G(y)$, if we put $y=g(e)$. As we assumed that $G(e)=Y$, we get the assertion.

Lemma 3. $H=G|Z$ is a group. $H(z)=Z$ for every $z \in Z$.

Proof:

According to [1], it is enough to prove that $H(z)=Z$ for every $z \in Z$. If $z' \in Z$, then $z' \in G(y)$, for any $y \in Y$, and therefore also $z' \in G(z)$. But $G(z)=H(z)$, as $G(z) \subset Z$, for Z is an invariant set.

Lemma 4. Let $g' \in H$. Then g' is either the identity map or g' has no fixed point.

Proof:

Let us suppose that $g'(z')=z'$, $z' \in Z$. By lemma 3, for arbitrary $z \in Z$ we can write $z=h'(z')$, where $h' \in H$. But then

$$g'(z)=g' \circ h'(z')=h' \circ g'(z')=h'(z')=z.$$

g' and h' commute, as they are the restrictions of commuting mappings. Hence g' is the identity mapping.

Lemma 5. Let Z have more than one point. Then there exists a mapping $g \in G$ such that g has no fixed point.

Proof:

Let $z_1 \in Z$. Then there exists $g_1 \in G$ such that $g_1(e)=z_1$. Evidently, $g_1(Y)=g_1(G(e))=G(g_1(e))=G(z_1) \subset Z$, as Z is an invariant set.
Hence, g_1 has no fixed point on $Y \setminus Z$. If $g_1|Z$ is not the identity map, then the lemma is proved, by lemma 3, and we can put $g=g_1$.
Let $g_1|Z=i|Z$, where i is the identity mapping. Then there exists $z_2 \in Z$, $z_1 \neq z_2$, and $g_2(z_1)=z_2$, $g_2 \in G$. Then $g_1 \circ g_2(Y) \subset Z$, and $g_1 \circ g_2(z_1) \neq z_1$. Putting $g=g_1 \circ g_2$, we get the assertion of the lemma.

Theorem 1. Let F be a commutative semigroup of continuous transformations of a topological space X, with F containing the identity map.

Then all the transformations which are elements of F have a common fixed point if and only if the orbit of some point is a compact space with f.p.p.

Proof:

If F has a common fixed point, then the orbit of this fixed point has the required properties.

Now, let $e \in X$ be the point such that F(e) is compact and has f.p.p. Let us denote $F(e)=Y$ and $F|Y=G$. Then, using the previous lemmas, we get immediately, that Z, as introduced in lemma 1, must have only one point. This point is a common fixed point of F.

Remark:

The assumption that F(e) has f.p.p. can be replaced by the assumption that every $f \in F$ has a fixed point in F(e).

We can apply the previous theorem to commutative topological semigroups. Every commutative topological semigroup (A;.) can be considered as a transformation semigroup of the space A into itself.

Moreover, if A is a topological space and (A;.) is a commutative semigroup, we shall say that (A;.) is a commutative semitopological semigroup, if for every net $a_\alpha \longrightarrow a$, $a_\alpha \in A$, $a \in A$, and for every $b \in A$

$$a_\alpha . b \longrightarrow a.b$$

is true.

Evidently every commutative topological semigroup is a commutative semitopological semigroup.

Applying theorem 1 to such semigroups we get:

Theorem 2. Let (A;.) be a commutative semitopological semigroup with the unity element. Let the topological space A be compact and have f.p.p. Then (A;.) has a zero.

Proof:

Let F consist of all mappings $f_a(b)=a.b$, $a \in A$. Then F and $X=A$ fulfil the assumptions of theorem 1. Therefore there exists $0 \in A$ such that

$$f_a(0)=0, \quad \text{for every } a \in A.$$

But that is the same as

$$a.0=0.$$

Reference

[1] Z. Hedrlín: On common fixed points of commutative mappings, Commentationes Mathematical Universitatis Carolinae, 2,4 (1961).

[2] A.D.Wallace: The structure of topological semigroup, Bulletin of the American Mathematical Society, 61,2 (1955).